Stability and Representation Dependence of the Quantum Skyrmion

A. Acus[1] E. Norvaišas[1] and D.O. Riska[2]

[1]*Institute of Theoretical Physics and Astronomy, Vilnius, 2600 Lithuania*
[2]*Department of Physics, University of Helsinki, 00014 Finland*

Abstract

A constructive realization of Skyrme's conjecture that an effective pion mass "may arise as a self consistent quantal effect" based on an ab initio quantum treatment of the Skyrme model is presented. In this quantum mechanical Skyrme model the spectrum of states with $I = J$, which appears in the collective quantization, terminates without any infinite tower of unphysical states. The termination point depends on the model parameters and the dimension of the $SU(2)$ representation. Representations, in which the nucleon and Δ_{33} resonance are the only stable states, exist. The model is developed for both irreducible and reducible representations of general dimension. States with spin larger than 1/2 are shown to be deformed. The representation dependence of the baryon observables is illustrated numerically.

1 Introduction

The modern view of Skyrme's topological soliton model of the baryons [1] is that represents a chiral symmetric effective mesonic representation of the approximately chiral symmetric QCD Lagrangian in the large N_c limit, in which the baryons have to be constructed as topological solitons [2]. The mass of the pion, which is the Goldstone boson of the spontaneously broken mode of chiral symmetry, is conventionally introduced through an external chiral symmetry breaking term in the Lagrangian density [3]. Skyrme, however, originally suggested a very different origin for the pion mass as "a self consistent quantal effect" [4].

We here provide a constructive realization of this conjecture, by demonstrating that in an ab initio quantum mechanical treatment of the Skyrme model Lagrangian, a term, which may be interpreted as an effective pion mass term, automatically arises. Moreover, with typical values for the parameters in the Skyrme Lagrangian, this mass term takes values close to that of the physical pion mass, although it is state dependent as a consequence of the quantization procedure. This effective pion mass may naturally be combined with the pion mass that may be introduced by adding an explicitly chiral symmetry breaking pion mass term to the Lagrangian density of the model.

The effective pion mass appears through an additional term in the Euler-Lagrange equation of the quantum Skyrme model, the asymptotic form of which is consistent with a partial conservation law for the axial current (PCAC). This term restores the stability of the soliton solutions, which is lost in a direct variational solution of the Skyrme model, when the rotational energy introduced by projection onto states of good spin and isospin is included [5, 6]. The quantum Skyrme model describes the baryon states with spin larger than $1/2$ as deformed, but only in representations with larger dimension than the fundamental one. Finally the spectrum of states with equal spin and isospin, which appears in the collective quantization, terminates in the quantum Skyrme model, and therefore unphysical states with large $I = J$ do not appear. The termination point in this spectrum depends on the parameters in the model as well as on the dimension of the representation employed. This representation dependence is a quantum effect, which

is absent in the classical Skyrme model [7].

The present work builds on the development of the ab initio quantized version of the Skyrme model in ref. [8], but goes beyond the treatment of the quantum corrections as perturbations of the classical Skyrmion. The method of ab initio treatment of the Skyrme model in $SU(2)$ representations of arbitrary dimension is that suggested in ref. [9] for the fundamental representation. The narrow stability constraints of the fundamental representation [10] are avoided by the development of the model in general representations. The ab initio quantum mechanical treatment differs from that of ref. [11], in which the systematic quantization of the Skyrme model was developed on the classical Hamiltonian of the model.

The present manuscript falls into 7 sections. In section 2 we review the basic formalism of the Skyrme model in a representation of arbitrary dimension. In section 3 we construct the quantum Skyrme model in representations of arbitrary dimension. In section 4 we derive the equations of motion and the associated effective pion mass. The Noether currents are derived in section 5. Numerical results for the baryon observables are presented in section 6. Section 7 contains a concluding discussion.

2 The classical skyrmion in a general representation.

The Lagrangian density of the Skyrme model [1] depends on a unitary field $U(\vec{r},t)$, which in a general reducible representation for the $SU(2)$ group may be expressed as direct sum of Wigner's D matrices

$$U(\vec{r},t)=\sum_k \oplus D^{j_k}[\vec{\alpha}(\vec{r},t)]. \tag{1}$$

Here $\vec{\alpha}$ represents a triplet of Euler angles $\alpha_1(\vec{r},t)$, $\alpha_2(\vec{r},t)$, $\alpha_3(\vec{r},t)$, which form the set of the dynamical variables. The D^{j_k} matrices have dimension

$(2j_k + 1) \times (2j_k + 1)$.

The Lagrangian density has the form [12]:

$$\mathcal{L}[U(\vec{r},t)] = -\frac{f_\pi^2}{4} Tr\{R_\mu R^\mu\} + \frac{1}{32e^2} Tr\{[R_\mu, R_\nu]^2\}, \qquad (2)$$

where the "right" current R_μ is defined as

$$R_\mu = (\partial_\mu U)U^\dagger, \qquad (3)$$

and f_π (the pion decay constant) and e are parameters.

The trace of a bilinear combination of generators $\hat{J}_a$ of the $SU(2)$ group depends on the representation as

$$Tr\langle j\cdot|\hat{J}_a \hat{J}_b|j\cdot\rangle = (-)^a \frac{1}{6} \sum_k j_k(j_k + 1)(2j_k + 1)\delta_{a,-b} \equiv (-)^a \frac{1}{4} N \delta_{a,-b}. \qquad (4)$$

The commutator relations for these generators are

$$[\hat{J}_a,\ \hat{J}_b] = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \end{bmatrix} \hat{J}_c. \qquad (5)$$

The factor in the square brackets on the r.h.s. is the Clebsch-Gordan coefficient $(1a1b|1c)$, in a more convenient notation. Here we have used the normalizations $\hat{J}_\pm = -J_{\pm 1}/\sqrt{2}$ and $\hat{J}_0 = -J_0/\sqrt{2}$.

In the classical treatment of the Skyrme model the Lagrangian density depends on the dimension of the irreducible representation only through the overall scalar factor N (4) [7]. In the case of a reducible representation this overall factor is a sum over separate factors for the different irreducible representations:

$$N = 2/3 \sum_k j_k(j_k + 1)(2j_k + 1). \qquad (6)$$

Because N is an overall factor the equations of motion for the dynamical variables $\vec{\alpha}$ are independent of the dimension of the representation.

The static "spherically symmetric" hedgehog ansatz in a general representation is invariant under the combined spatial and isospin rotations:

$$i\,[\vec{r} \times \nabla]_a\, U(\vec{r}) + \sqrt{2}\left[\widehat{J}_a, U(\vec{r})\right] = 0. \qquad (7)$$

Here circular components are used. The solution of (7) is the generalization of the usual hedgehog ansatz

$$e^{i\vec{\tau}\cdot\vec{r}F(r)} \Longrightarrow U_0(\vec{r}) = exp\{-i\sqrt{2}\hat{J}_a \cdot \hat{r}^a F(r)\}, \tag{8}$$

where $\hat{r}$ is the unit vector expressed in terms of circular components.

With the hedgehog ansatz (8) the Lagrangian density (2) reduces to the following simple form

$$\begin{aligned} \mathcal{L}[F(r)] &= -N\mathcal{M}(F(r)) = -N\Big\{\frac{f_\pi^2}{2}\Big(F'^2 + \frac{2}{r^2}\sin^2 F\Big) \\ &\quad + \frac{1}{8e^2}\frac{\sin^2 F}{r^2}\Big(2F'^2 + \frac{\sin^2 F}{r^2}\Big)\Big\}. \end{aligned} \tag{9}$$

The requirement that the soliton mass be stationary yields the same differential equation for the chiral angle $F(r)$ as in [12]:

$$F'' + 2F''\frac{\sin^2 F}{\tilde{r}^2} + F'^2\frac{\sin 2F}{\tilde{r}^2} + \frac{2}{\tilde{r}^2}F' - \frac{\sin 2F}{\tilde{r}^2} - \frac{\sin 2F \sin^2 F}{\tilde{r}^4} = 0. \tag{10}$$

Here the dimensionless variable $\tilde{r}$ is defined as $\tilde{r} = ef_\pi r$. The boundary conditions for solitons with unit baryon number are $F(0) = \pi$, $F(\infty) = 0$. The classical soliton is obviously independent of the the representation.

After the renormalization the hedgehog mass in any representation has the form:

$$\begin{aligned} M(F) &= \int \mathrm{d}^3 r\mathcal{M}(F(r)) = \frac{f_\pi}{e}\widetilde{M}(F) = \\ & 2\pi\frac{f_\pi}{e}\int \mathrm{d}\tilde{r}\tilde{r}^2\left[F'^2 + \frac{\sin^2 F}{\tilde{r}^2}\left(2 + 2F'^2 + \frac{\sin^2 F}{\tilde{r}^2}\right)\right]. \end{aligned} \tag{11}$$

For the hedgehog solution the baryon density takes the form

$$B^0 = \frac{\epsilon^{0\nu\beta\gamma}}{24\pi^2 N} Tr\, R_\nu\, R_\beta\, R_\gamma = -\frac{1}{2\pi^2}\frac{\sin^2 F}{r^2}F'. \tag{12}$$

The renormalization factor N ensures that the lowest nonvanishing baryon number is $B = 1$ for the hedgehog an in arbitrary representation.

3 Quantization in the collective coordinate approach

The quantization of the Skyrme model in a general dimension [8] can be achieved by means of collective rotational coordinates that separate the variables which depend on the time and spatial coordinates [12]:

$$U(\vec{r},\vec{q}(t)) = A\left(\vec{q}(t)\right) U_0(\vec{r}) A^{\dagger}\left(\vec{q}(t)\right). \tag{13}$$

The set of three real, independent parameters $\vec{q}(t) = (q^1(t), q^2(t), q^3(t))$ are quantum variables (Euler angles representing rotations of the Skyrmion). In a general representation the unconstrained variables $\vec{q}(t)$ are more convenient than the four constrained Euler-Rodrigues parameters with the constraint used in [12] . When the the Skyrme Lagrangian (2) is considered quantum mechanically ab initio the generalized coordinates $\vec{q}(t)$ and velocities $\vec{\dot{q}}(t)$ satisfy the commutation relations [9]

$$[\dot{q}^a,\, q^b] = -if^{ab}(\vec{q}). \tag{14}$$

Here the tensor $f^{ab}(\vec{q})$ is a function of generalized coordinates $\vec{q}$ only, the explicit form of which is determined after the quantization condition has been imposed. The tensor f^{ab} is symmetric with respect to interchange of the indices a and b as a consequence of the relation $[q^a,\, q^b] = 0$. The commutation relation between a generalized velocity component $\dot{q}^a$ and arbitrary function $G(\vec{q})$ is given by

$$[\dot{q}_a,\, G(\vec{q})] = -i\sum_r f^{ar}(\vec{q})\frac{\partial}{\partial q^r}G(\vec{q}). \tag{15}$$

Here Weyl ordering of the operators has been employed:

$$\partial_0 G(q) = \frac{1}{2}\{\dot{q}^{\alpha}, \frac{\partial}{\partial q^{\alpha}}G(q)\}. \tag{16}$$

With this choice of operator ordering no further ordering ambiguity appears.

After making the substitution (13) into the Lagrangian density (2) the dependence of Lagrangian on the generalized velocities can be expressed as

$$L(\vec{q},\vec{\dot{q}},F) \;=\; \frac{1}{N}\int \mathrm{d}^3 r\mathcal{L}(\vec{r},\vec{q}(t),F(r)) =$$

$$-\frac{1}{4}a(F)\dot{q}^{\alpha}g(\vec{q})_{\alpha\beta}\dot{q}^{\beta}+\mathcal{O}(\dot{q}^{0}). \tag{17}$$

Here the soliton moment of inertia $a(F)$ is given as

$$\begin{aligned} a(F) &= \int \mathrm{d}^3 r \mathcal{A}(F(r)) = \frac{1}{e^3 f_\pi}\tilde{a}(F) = \\ & \frac{1}{e^3 f_\pi}\frac{8\pi}{3}\int \mathrm{d}\tilde{r}\tilde{r}^2 \sin^2 F\left[1+F'^2+\frac{\sin^2 F}{\tilde{r}^2}\right], \end{aligned} \tag{18}$$

where $\mathcal{A}(F(r))$ is the moment of inertia density.

The 3×3 metric tensor $g(\vec{q})_{\alpha\beta}$ is defined as the scalar product of a set of functions $C_\alpha^{(m)}(\vec{q})$ [7]

$$\begin{aligned} g(\vec{q})_{\alpha\beta} &= \sum_m (-)^m C_\alpha^{(m)} C_\beta^{(-m)} = \sum_m (-)^m C_\alpha'^{(m)} C_\beta'^{(-m)} = \\ & -2\delta_{\alpha\beta} - 2(\delta_{\alpha 1}\delta_{\beta 3}+\delta_{\alpha 3}\delta_{\beta 1})\cos q^2. \end{aligned} \tag{19}$$

Here the functions $C_\alpha'^{(m)}$ are defined in [8] and related with $C_\alpha^{(m)}$ as

$$C_\alpha^{(m)}(\vec{q}) = \sum_m D^1_{m,m'}(\vec{q}) C_\alpha'^{(m)}(\vec{q}). \tag{20}$$

The canonical momentum p_α, which is conjugate to q^α, is defined as

$$p_\alpha(\vec{q},\vec{\dot{q}},F) = \frac{\partial L(\vec{q},\vec{\dot{q}},F)}{\partial \dot{q}^\alpha} = -\frac{1}{4}a(F)\{\dot{q}^\beta, g(\vec{q})_{\beta\alpha}\}, \tag{21}$$

where the curly bracket denotes an anticommutator. The canonical commutation relations

$$\left[p_\alpha(\vec{q},\vec{\dot{q}},F), q^\beta\right] = -i\delta_{\alpha\beta}, \tag{22}$$

then yield the following explicit form for the functions $f^{ab}(\vec{q})$:

$$f^{ab}(\vec{q}) = -\frac{2}{a(F)} g^{-1}_{\alpha\beta}(\vec{q}). \tag{23}$$

Because of the nonlinearity of the Skyrme model the canonical momenta defined in this way do not necessarily satisfy the relation $[p_\alpha, p_\beta] = 0$. As shown in [9], there exists a local transformation of the set of variables $\vec{q}$,

which makes it possible to satisfy these relations. Following Fujii et al. [9] we define an angular momentum operator:

$$\hat{J}'_a = -\frac{i}{2}\left\{p_r, C'^r_{(-a)}(\vec{q})\right\} = (-)^a \frac{ia(F)}{4}\left\{\dot{q}^r, C'^{(-a)}_r(\vec{q})\right\}, \tag{24}$$

which satisfies the commutation relation (5). It is readily verified that the operator $\hat{J}'_a$ is a $D^j(\vec{q})$ "right rotation" generator. The explicit form for the Lagrangian of the consistently quantized Skyrme model now takes the form:

$$\begin{aligned} L(\vec{q}, \dot{\vec{q}}, F) &= -M(F) - \Delta M_{\Sigma j}(F) + \frac{1}{a(F)}\hat{J'}^2 = \\ & -M(F) - \Delta M_{\Sigma j}(F) + \frac{1}{a(F)}\hat{J}^2, \end{aligned} \tag{25}$$

where

$$\Delta M_{\Sigma j}(F) = \int \mathrm{d}^3 r \Delta\mathcal{M}_{\Sigma j}(F(r)) = e^3 f_\pi \cdot \Delta\widetilde{M}_{\Sigma j}(F) = -e^3 f_\pi \frac{2\pi}{15\tilde{a}^2(F)}$$

$$\times \int \mathrm{d}\tilde{r}\tilde{r}^2 \sin^2 F\Big\{15 + 4d_2 \sin^2 F(1 - F'^2) + 2d_3\frac{\sin^2 F}{\tilde{r}^2} + 2d_1 F'^2\Big\}. \tag{26}$$

The coefficients d_j in these expressions are given as

$$d_1 = \frac{1}{N}\sum_k j_k(j_k+1)(2j_k+1)\left[8j_k(j_k+1) - 1\right], \tag{27}$$

$$d_2 = \frac{1}{N}\sum_k j_k(j_k+1)(2j_k+1)(2j_k-1)(2j_k+3), \tag{28}$$

$$d_3 = \frac{1}{N}\sum_k j_k(j_k+1)(2j_k+1)\left[2j_k(j_k+1) + 1\right]. \tag{29}$$

The corresponding Hamilton operator is

$$H_j(F) = M(F) + \Delta M_j(F) + \frac{1}{a(F)}\hat{J'}^2 = M(F) + \Delta M_j(F) + \frac{1}{a(F)}\hat{J}^2. \tag{30}$$

This result differs from the semiclassical one in the appearance of the negative quantum correction $\Delta M_j(F)$ [9], which depends on the dimension of the representation of the $SU(2)$ group [8].

For the Hamiltonian (30) the normalized state vectors with fixed spin and isospin ℓ are

$$\left| \begin{array}{c} \ell \\ m, m' \end{array} \right\rangle = \frac{\sqrt{2\ell+1}}{4\pi} D^{\ell}_{m,m'}(\vec{q}) \, |0\rangle \,, \tag{31}$$

with the eigenvalues

$$H(j,\ell,F) = M(F) + \Delta M_j(F) + \frac{\ell(\ell+1)}{2a(F)}. \tag{32}$$

Substitution of the rotated hedgehog (13) into the Lagrangian density (2) yields the following expression for the Lagrangian density for the quantum Skyrme model in a general reducible representation:

$$\mathcal{L}(\vec{r}, \vec{q}(t), F(r)) = \frac{3\mathcal{A}(F(r))}{2\, a^2(F)} \Big(\widehat{J}'^2 - (\widehat{J}' \cdot \hat{r})(\widehat{J}' \cdot \hat{r}) \Big) - \Delta\mathcal{M}_{\Sigma j}(F(r)) - \mathcal{M}(F(r)). \tag{33}$$

The angular momentum operator on the r.h.s. of (33) can be separated into scalar and tensor terms in the usual way:

$$\begin{aligned} \widehat{J}'^2 - (\widehat{J}' \cdot \hat{r})(\widehat{J}' \cdot \hat{r}) \;\; &= \;\; \frac{2}{3}\widehat{J}'^2 - \frac{4\pi}{3} Y^*_{2,m+m'}(\vartheta,\varphi) \\ &\quad \times \left[\begin{array}{ccc} 1 & 1 & 2 \\ m & m' & m+m' \end{array} \right] \widehat{J}'_m \widehat{J}'_{m'}, \end{aligned} \tag{34}$$

where $Y_{l,m}(\vartheta,\varphi)$ is a spherical harmonic.

The volume integral of the Lagrangian density (33) gives the Lagrangian (25). In the fundamental representation, for which $j = 1/2$, the second rank tensor part of (34) vanishes. This implies that the quadrupole moment of the Δ_{33} resonance cannot be described with the fundamental representation.

The Hamiltonian density, which corresponds to the quantum Lagrangian (33) has the following matrix elements for baryon states with spin and isospin $\ell > 1/2$:

$$\begin{aligned} \left\langle \begin{array}{c} \ell \\ m_t m_s \end{array} \right| \mathcal{H}(\vec{r}, \vec{q}(t), F(r)) \left| \begin{array}{c} \ell \\ m_t m_s \end{array} \right\rangle &= \mathcal{M}(F(r)) + \Delta\mathcal{M}_{\Sigma j}(F(r)) \\ &+ \frac{\mathcal{A}(F(r))}{2\, a^2(F)} \left(\ell(\ell+1) - \sqrt{\frac{2}{3}}\pi [3m_s^2 - \ell(\ell+1)] Y_{2,0}(\vartheta,\varphi) \right). \end{aligned} \tag{35}$$

For nucleons $\ell = 1/2$ the dependence on angles are absent and the quantum skyrmion is therefore spherically symmetric as required.

4 The Chiral Angle and the Pion Mass

The $I = J = \ell = 1/2$ and $I = J = \ell = 3/2$ skyrmions are to be identified with the nucleons and the Δ_{33} resonances. Minimization of the classical expression for the mass $M(F)$ (11) leads to the conventional differential equation for the chiral angle $F(r)$ (10) according to which $F(r)$ falls as $1/r^2$ at large distances.

In the semiclassical approach the quantum mass term $\Delta M_{\Sigma j}$ is absent from the mass expression (30). Its absence has the consequence that variation of the truncated quantum mass expression yields no stable solution [5, 6]. The semiclassical approach describes the skyrmion as a "rotating" rigid-body with fixed $F(r)$ [12]. In contrast the full energy expression (30) that is obtained in the consistent canonical quantization procedure in collective coordinates approach gives stable solutions.

Minimization of the quantum mass expression (30), leads to the following integro-differential equation for the chiral angle $F(r)$:

$$\begin{aligned}
&F''\Big[-2\tilde{r}^2 - 4\sin^2 F + \frac{e^4\tilde{r}^2\sin^2 F}{15\tilde{a}^2(F)}\Big\{80\tilde{a}(F)\Delta\widetilde{M}_{\Sigma j}(F) + 20\ell(\ell+1) \\
&+4d_1 - 8d_2\sin^2 F\Big\}\Big] + F'^2\Big[-2\sin 2F + \frac{e^4\tilde{r}^2\sin 2F}{15\tilde{a}^2(F)}\Big\{40\tilde{a}(F)\Delta\widetilde{M}_{\Sigma j}(F) \\
&+10\ell(\ell+1) + 2d_1 - 8d_2\sin^2 F\Big\}\Big] + F'\Big[-4\tilde{r} + \frac{e^4\tilde{r}\sin^2 F}{15\tilde{a}^2(F)} \\
&\times\Big\{160\tilde{a}(F)\Delta\widetilde{M}_{\Sigma j}(F) + 40\ell(\ell+1) + 8d_1 - 16d_2\sin^2 F\Big\}\Big] + \sin 2F\Big[2 \\
&+2\frac{\sin^2 F}{\tilde{r}^2} - \frac{e^4}{15\tilde{a}^2(F)}\Big\{\Big(40\tilde{a}(F)\Delta\widetilde{M}_{\Sigma j}(F) + 10\ell(\ell+1)\Big)\Big(\tilde{r}^2 + 2\sin^2 F\Big) \\
&+15\tilde{r}^2 + 4d_3\sin^2 F + 8d_2\tilde{r}^2\sin^2 F\Big\}\Big] = 0, \qquad (36)
\end{aligned}$$

with the usual boundary conditions $F(0)=\pi$ and $F(\infty)=0$.
The state dependence of this equation of motion is a direct consequence of the fact that quantization preceeded variation (cf. ref. [13]).

At large distances this equation reduces to the asymptotic form:

$$\tilde{r}^2 F'' + 2\tilde{r}F' - (2+\tilde{m}_\pi^2\tilde{r}^2)F = 0, \tag{37}$$

where the quantity $\tilde{m}_\pi^2$ is defined as

$$\tilde{m}_\pi^2 = -\frac{e^4}{3\tilde{a}(F)}\left\{8\Delta\widetilde{M}_{\Sigma j}(F) + \frac{2\ell(\ell+1)+3}{\tilde{a}(F)}\right\}. \tag{38}$$

The corresponding asymptotic solution takes the form

$$F(\tilde{r}) = k\left(\frac{\tilde{m}_\pi}{\tilde{r}} + \frac{1}{\tilde{r}^2}\right)\exp(-\tilde{m}_\pi\tilde{r}). \tag{39}$$

The requirement of stability of the quantum skyrmion is that the integrals (11), (18) and (26) converge. This requirement is satisfied only if $\tilde{m}_\pi^2 > 0$. For that the presence of the negative quantum correction $\Delta M_j(F)$ is necessary. It is the absence of this term, which leads to the instability of the skyrmion in the semiclassical approach [6]. Note that in the quantum treatment the chiral angle has the asymptotic Yukawa behaviour (39) even in the chiral limit [9]. The positive quantity $m_\pi = e f_\pi \tilde{m}_\pi$ should obviously be interpreted as an effective mass for the pion field in the Skyrmion mass.

In contrast to the classical skyrmion the stability of the quantum mechanical skyrmion depends on the Lagrangian parameter values f_π and e [10]. Moreover positivity of the pion mass (38), can obviously only be achieved for states with sufficiently small values of spin ℓ. This implies that the spectrum of states with equal spin and isospin will necessarily terminate at some finite value of the spin quantum number. As the negative quantum mass correction $\Delta M_{\Sigma j}$ in the expression (26) grows in magnitude with the dimension of the representation, it is always possible to find a representation in which the nucleon and the Δ_{33} resonance are the only stable particles, as required by experiment. This argument is more general than the method of self consistent dynamical truncation of the spectrum suggested in ref. [14].

5 The Noether currents

The Lagrangian density of the Skyrme model is invariant under left and right transformations of the unitary field U. The corresponding Noether currents can be expressed in terms of the collective coordinates (13). The vector and axial Noether currents that are associated with the transformations,

$$U(x) \xrightarrow{V(A)} \left(1 - i2\sqrt{2}\omega^a \hat{J}_a\right) U(x) \left(1 + (-)i2\sqrt{2}\omega^a \hat{J}_a\right), \tag{40}$$

are nevertheless simpler and directly related to physical observables. The factor $-2\sqrt{2}$ before generators is needed in the case $j = 1/2$ to match the transformation (40) with the infinitesimal transformation in [12]. The Noether currents are operators that depend on the generalized collective coordinates $\vec{q}$ and the generalized angular momentum operators $\hat{J}'_a$ (24).

The explicit expression for the spatial components of the vector current density is

$$\begin{aligned} \hat{V}_b^a \quad = \quad & \frac{\partial \mathcal{L}_V}{\partial\left(\nabla^b \omega_a\right)} = 2\sqrt{2}\frac{\sin^2 F}{r}\left(i\Big\{f_\pi^2 + \frac{1}{e^2}\Big(F'^2\right. \\ & + \frac{\sin^2 F}{r^2} - \frac{2d_2+5}{4\cdot 5\cdot a^2(F)}\sin^2 F\Big)\Big\} \left[\begin{array}{ccc} 1 & 1 & 1 \\ u & s & b \end{array}\right] D^1_{a,s}(\vec{q})\hat{x}_u \\ & - \frac{\sin^2 F}{\sqrt{2}\cdot e^2 \cdot a^2(F)}(-)^s\Big\{\left[\hat{J}'\times\hat{r}\right]_{-s} D^1_{a,s}(\vec{q})\left[\left[\hat{J}'\times\hat{r}\right]\times\hat{r}\right]_b \\ & \left. + \left[\left[\hat{J}'\times\hat{r}\right]\times\hat{r}\right]_b D^1_{a,s}(\vec{q})\left[\hat{J}'\times\hat{r}\right]_{-s}\Big\}\right). \end{aligned} \tag{41}$$

Here ∇^k is a circular component of the gradient operator. The indices a and b denote isospin and spin components. The time (charge) component of the vector current density has the expression

$$\begin{aligned} \hat{V}_t^a \quad = \quad & \frac{\partial \mathcal{L}_V}{\partial\left(\partial_0 \omega_a\right)} = \frac{2\sqrt{2}}{a(F)}\sin^2 F\left[f_\pi^2 + \frac{1}{e^2}\left(F'^2 + \frac{\sin^2 F}{r^2}\right)\right] \\ & \times(-)^s\left\{D^1_{a,-s}(\vec{q})\hat{J}'_s - D^1_{a,-s}(\vec{q})\hat{x}_s(\hat{J}'\cdot\hat{r})\right\}. \end{aligned} \tag{42}$$

The explicit expression for the axial current density takes the form

$$\hat{A}^a_b = \frac{\partial \mathcal{L}_A}{\partial \left(\nabla^b \omega_a\right)} = \Bigg(\Bigg\{ f_\pi^2 \frac{\sin 2F}{r} + \frac{1}{e^2}\frac{\sin 2F}{r}\Big(F'^2 + \frac{\sin^2 F}{r^2} - \frac{\sin^2 F}{4\cdot a^2(F)}\Big)\Bigg\}$$
$$\times D^1_{a,b}(\vec{q}) + \Bigg\{ f_\pi^2 \Big(2F' - \frac{\sin 2F}{r}\Big) - \frac{1}{e^2}\Big(F'^2 \frac{\sin 2F}{r} - 4F' \frac{\sin^2 F}{r^2}$$
$$+ \frac{\sin^2 F \sin 2F}{r^3} - \frac{\sin^2 F \sin 2F}{4\cdot a^2(F)\cdot r}\Big)\Bigg\}(-)^s D^1_{a,s}(\vec{q})\hat{x}_{-s}\hat{x}_b - \frac{2F' \sin^2 F}{e^2 \cdot a^2(F)}$$
$$\times (-)^s \left\{ D^1_{a,s}(\vec{q})\hat{x}_{-s}\hat{J}'^2 + \hat{J}'^2 D^1_{a,s}(\vec{q})\hat{x}_{-s} - 2D^1_{a,s}(\vec{q})\hat{x}_{-s}(\hat{J}'\cdot\hat{r})(\hat{J}'\cdot\hat{r})\right\}\hat{r}_b$$
$$- \frac{\sin^2 F \sin 2F}{e^2\cdot a^2(F)\cdot r}(-)^s \left\{ \left[\left[\hat{J}'\times\hat{r}\right]\times\hat{r}\right]_{-s} D^1_{a,s}(\vec{q}) \left[\left[\hat{J}'\times\hat{r}\right]\times\hat{r}\right]_b\right.$$
$$\left. + \left[\left[\hat{J}'\times\hat{r}\right]\times\hat{r}\right]_b D^1_{a,s}(\vec{q}) \left[\left[\hat{J}'\times\hat{r}\right]\times\hat{r}\right]_{-s}\right\}\Bigg). \tag{43}$$

The operators (41)-(43) are well defined for all representations j of the classical soliton and for fixed spin and isospin ℓ of the quantum skyrmion. The new terms, which are absent in the corresponding semiclassical expression, are those that have the factor $a^2(F)$ in the denominator.

The conserved topological current density in Skyrme model is the baryon current density, the components of which are

$$\mathcal{B}_a(\vec{r}, F(r)) = \frac{1}{\sqrt{2}\pi^2 a(F)\, r}\sin^2 F \cdot F' \left[\hat{J}'\times\hat{r}\right]_a. \tag{44}$$

The matrix elements of the third component of the isoscalar magnetic moment operator have the form

$$\left\langle \begin{array}{c} \ell \\ m_t m_s \end{array}\left| [\mu_{I=0}]_3 \right| \begin{array}{c} \ell \\ m_t m_s \end{array}\right\rangle = \left\langle \begin{array}{c} \ell \\ m_t m_s \end{array}\left| \frac{1}{2}\int d^3 x r\, [\hat{r}\times\mathcal{B}]_0 \right| \begin{array}{c} \ell \\ m_t m_s \end{array}\right\rangle =$$
$$\frac{[\ell(\ell+1)]^{1/2}\, e}{3\cdot\tilde{a}(F) f_\pi}\left\langle \tilde{r}^2_{I=0}\right\rangle \left[\begin{array}{ccc} \ell & 1 & \ell \\ m_s & 0 & m_s \end{array}\right]. \tag{45}$$

Here the isoscalar mean square radius is given as

$$\left\langle r^2_{E,I=0}\right\rangle = \frac{1}{e^2 f_\pi^2}\left\langle \tilde{r}^2_{I=0}\right\rangle = -\frac{1}{e^2 f_\pi^2}\frac{2}{\pi}\int \tilde{r}^2 \sin^2 F\cdot F' d\tilde{r}, \tag{46}$$

and the quantity $\tilde{a}$ is defined in eq. (18).

The matrix elements of the third component of the isovector part of magnetic moment operator that is obtained from the vector current density (41) have the form

$$\left\langle \begin{array}{c} \ell \\ m_t m_s \end{array} \left| [\mu_{I=1}]_3 \right| \begin{array}{c} \ell \\ m_t m_s \end{array} \right\rangle = \left\langle \begin{array}{c} \ell \\ m_t m_s \end{array} \left| \frac{1}{2} \int d^3x \cdot r \left[\hat{r} \times \hat{V}^3 \right]_0 \right| \begin{array}{c} \ell \\ m_t m_s \end{array} \right\rangle =$$
$$\left[\frac{\tilde{a}(F)}{e^3 \cdot f_\pi} + \frac{8\pi \cdot e}{3 \cdot f_\pi \cdot \tilde{a}^2(F)} \int d\tilde{r} \cdot \tilde{r}^2 \sin^4 F \left(1 - \frac{d_2}{2 \cdot 5} - \frac{\ell(\ell+1)}{3} \right.\right.$$
$$\left.\left. + \frac{(-)^{2\ell}}{2} \left[\frac{5\ell(\ell+1)(2\ell-1)(2\ell+1)(2\ell+3)}{2 \cdot 3} \right]^{1/2} \left\{ \begin{array}{ccc} 1 & 2 & 1 \\ \ell & \ell & \ell \end{array} \right\} \right) \right]$$
$$\times \left[\begin{array}{ccc} \ell & 1 & \ell \\ m_s & 0 & m_s \end{array} \right] \left[\begin{array}{ccc} \ell & 1 & \ell \\ m_t & 0 & m_t \end{array} \right]. \tag{47}$$

Here the symbol in the curly brackets is a $6j$ coefficient.

The volume integral of the axial current density (42) yields the axial coupling constant g_A as

$$g_A = -3 \left\langle \begin{array}{c} 1/2 \\ 1/2, 1/2 \end{array} \left| \int d^3x A_0^0 \right| \begin{array}{c} 1/2 \\ 1/2, 1/2 \end{array} \right\rangle = \frac{1}{e^2} \tilde{g}_1(F) - \frac{\pi^2 e^2}{3 \cdot \tilde{a}^2(F)} \left\langle \tilde{r}^2_{I=0} \right\rangle, \tag{48}$$

where

$$\begin{aligned} \tilde{g}_1(F) \quad = \quad & \frac{4\pi}{3} \int d\tilde{r} (\tilde{r}^2 F' + \tilde{r} \sin 2F + \tilde{r} \sin 2F \cdot F'^2 \\ & + 2 \sin^2 F \cdot F' + \frac{\sin^2 F}{\tilde{r}} \sin 2F). \end{aligned} \tag{49}$$

For nucleons the the isovector charge mean square radius becomes

$$\left\langle r^2_{E,I=1} \right\rangle = \frac{1}{e^2 f_\pi^2} \left\langle \tilde{r}^2_{E,I=0} \right\rangle = \frac{1}{e^2 f_\pi^2} \frac{\int d\tilde{r} \tilde{r}^4 \sin^2 F \left[1 + F'^2 + \frac{\sin^2 F}{\tilde{r}} \right]}{\int d\tilde{r} \tilde{r}^2 \sin^2 F \left[1 + F'^2 + \frac{\sin^2 F}{\tilde{r}} \right]}. \tag{50}$$

The isoscalar magnetic mean square radius has the expression

$$\left\langle r^2_{M,I=0} \right\rangle = \frac{1}{e^2 f_\pi^2} \left\langle \tilde{r}^2_{M,I=0} \right\rangle = -\frac{1}{e^2 f_\pi^2} \frac{2}{\pi} \frac{\int d\tilde{r} \tilde{r}^4 \sin^2 F \cdot F'}{\int d\tilde{r} \tilde{r}^2 \sin^2 F \cdot F'}, \tag{51}$$

and the isovector magnetic mean square radius the expression

$$\left\langle r^2_{M,I=1}\right\rangle = \frac{1}{e^2 f_\pi^2}\left\langle \tilde{r}^2_{M,I=0}\right\rangle = \frac{1}{e^2 f_\pi^2}\frac{\int d\tilde{r}\tilde{r}^4 \sin^2 F\left[1+F'^2+\frac{\sin^2 F}{\tilde{r}}+\frac{e^2\sin^2 F}{\tilde{a}^2(F)}\left(\frac{3}{4}-\frac{d_2}{10}\right)\right]}{\int d\tilde{r}\tilde{r}^2 \sin^2 F\left[1+F'^2+\frac{\sin^2 F}{\tilde{r}}+\frac{e^2\sin^2 F}{\tilde{a}^2(F)}\left(\frac{3}{4}-\frac{d_2}{10}\right)\right]}. \tag{52}$$

The matrix element of the divergence of the vector current (41) vanishes:

$$\left\langle \begin{array}{c}\ell \\ m_t m_s\end{array}\middle| \nabla^b \widehat{V}^a_b \middle| \begin{array}{c}\ell \\ m_t m_s\end{array}\right\rangle = 0. \tag{53}$$

The the asymptotic equation of motion (37), valid for large r, is recovered if matrix element of the divergence of the axial current (43) for the proton is taken to be

$$\left\langle \begin{array}{c}\ell \\ m_t m_s\end{array}\middle| \nabla^b \widehat{A}^a_b \middle| \begin{array}{c}\ell \\ m_t m_s\end{array}\right\rangle = f_\pi^2 m_\pi^2 F(r). \tag{54}$$

This is the usual equation for a "partially conserved axial current" (PCAC), and supports the interpretation of m_π as an effective pion mass.

6 Numerical Results

The equation of motion for the quantum Skyrme model (36) depends – in contrast to the classical case – on the parameter e and representation. Moreover the differential equation for the chiral angle is state dependent.

For nucleons ($\ell = 1/2$) solutions for the chiral angle, which describe stable solitons with spin 1/2, exist when $e < 7.5$. The largest value of e, for which stable solutions are obtained, decreases with increasing dimensionality of the representation. For Δ_{33} resonances ($\ell = 3/2$) there are no stable soliton solutions in the fundamental representation, nor in the representation with $j = 1$ in the quantum Skyrme model. In the representations with $j = 3/2$ and 2 there are only stable soliton solutions for baryons with spin $\ell = 1/2$ and 3/2. A dimension with $j = 5/2$ allows stable solitons with spin $\ell = 1/2$, 3/2 and 5/2, and therefore appears to be empirically contraindicated.

The two parameters of the model, f_π and e, may be determined in the usual way by fitting two empirical baryon observables.. The procedure adopted here was to first determine these two parameters by using the chiral angle of the classical Skyrme model, which is independent of both the model parameters and the dimension of the representation [7], by a fit to the nucleon mass (939 MeV) and its isoscalar radius (0.72 fm) for different values of the dimension j of the representation. These parameters where then used in a numerical solution of the equation (36). That solution was subsequently used to determine new values of f_π and e. This procedure was iterated until a converged solution was obtained. The numerical results are shown in Table 1. For the irreducible representation with $j = 1$ the proton magnetic moment calculated in this way is within 10% of the empirical value. The calculated values of both the neutron magnetic moment and the axial coupling constant agree with the corresponding empirical values to within 1%. The Δ_{33} resonance observables for different representations as obtained with fixed values for f_π and e are presented in Table 2.

7 Discussion

There are two main aspects of the quantum corrections to the Skyrme model based description of the baryons. One is the treatment of the dynamical field variables of the Lagrangian density as quantum mechanical variables ab initio. This very likely formed the basis for Skyrme's conjecture for the origin of the pion mass [4]. The development of the ab initio quantum mechanical treatment of the model was pioneered in ref. [9], and was developed above to realize Skyrme's conjecture constructively. The other main aspect is the treatment of quantum fluctuations of the pion field as loop corrections [15].

Both types of quantum effects lead to substantial modifications of the phenomenological description of the baryons based on the Skyrme model. Both also lead to negative quantum corrections to the Skyrmion mass. In the present work this negative mass correction was shown to imply a positive effective mass for the pion field in the Skyrmion, and to stable variational solutions for the quantum soliton.

The spectrum of states with $I = J$ terminates in the quantum Skyrme model, because the effective pion mass becomes negative for sufficiently large spin. Moreover it describes the states with spin larger than 1/2 as deformed. That deformation is inherent in a Skyrmion model with a terminating spectrum has also been been noted in [16], although in that work the absence of states with large spin and isospin was achieved by associating very large decay widths with those states.

The systematic quantum mechanical treatment of the Skyrmion was shown to imply the need to employ representations of larger dimension than the fundamental one for the description of the nucleon and the Δ_{33} resonance as stable solitons with spin and isospin 1/2 and 3/2 respectively in the same representation. The quantum treatment implies that the tower of states with $I = J$ terminates, and that there therefore is no infinite tower of unphysical states as in the semiclassical approach.

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Table 1: The predicted static nucleon observables in different representations with fixed empirical values for the isoscalar radius 0.72 fm. and nucleon mass 939 MeV.

j	$1/2$	1	$3/2$	$5/2$	$1\oplus\frac{1}{2}\oplus\frac{1}{2}$	Exp.
m_N	input	input	input	input	input	939 MeV
f_π	59.8	58.5	57.7	56.6	58.8	93 MeV
e	4.46	4.15	3.86	3.41	4.24	
μ_p	2.60	2.52	2.51	2.52	2.53	2.79
μ_n	-2.01	-1.93	-1.97	-2.05	-1.93	-1.91
g_A	1.20	1.25	1.33	1.52	1.23	1.26
m_π	79.5	180.	248.	336.	155.	138MeV
$\sqrt{\langle r_E^2\rangle}_{I=0}$	input	input	input	input	input	0.72fm
$\sqrt{\langle r_E^2\rangle}_{I=1}$	1.33	1.03	0.97	0.93	1.07	0.88fm
$\sqrt{\langle r_M^2\rangle}_{I=0}$	1.05	1.01	1.00	1.00	1.01	0.81fm
$\sqrt{\langle r_M^2\rangle}_{I=1}$	1.32	1.03	0.97	0.93	1.07	0.80fm

Table 2: The predicted static Δ_{33}-resonances observables in different representations with fixed values for the parameters $e = 4.15$ and $f_\pi = 58.5$ (determined by a fit to the nucleon observables $m_N = 939$, $\langle r^2\rangle_{I=0}^{1/2} = 0.72$, in a representation with $j = 1$).

j	$\frac{3}{2}\oplus 1\oplus\frac{1}{2}$	$\frac{3}{2}$	2	Exp.
m_Δ	1055.	1029.	910.	1232 MeV
$\mu_{\Delta^{++}}$	7.38	6.40	4.20	$3.7-7.5$
$\mu_{\Delta^{+}}$	3.02	2.73	2.01	?
$\mu_{\Delta^{0}}$	-1.33	-0.94	-0.19	?
$\mu_{\Delta^{-}}$	-5.69	-4.61	-2.38	?
$\sqrt{\langle r_E^2\rangle}_{I=0}$	0.91	0.87	0.72	?